Mathematical aspects of intertwining operators: the role of Riesz bases

2010 J. Phys. A: Math. Theor. 43 175203

(http://iopscience.iop.org/1751-8121/43/17/175203)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 175203 (11pp) doi:10.1088/1751-8113/43/17/175203

Mathematical aspects of intertwining operators: the role of Riesz bases

F Bagarello

Dipartimento di Metodi e Modelli Matematici, Facoltà di Ingegneria, Università di Palermo, I-90128 Palermo, Italy

E-mail: bagarell@unipa.it

Received 29 December 2009, in final form 3 March 2010
Published 13 April 2010
Online at stacks.iop.org/JPhysA/43/175203

Abstract
In this paper we continue our analysis of intertwining relations for both self-adjoint and not self-adjoint operators. In particular, in this last situation, we discuss the connection with pseudo-Hermitian quantum mechanics and the role of Riesz bases.

PACS numbers: 45.20.Jj, 47.10.Df, 03.65.−w

1. Introduction

In a series of recent papers [1–4], we have proposed a new technique which produces, given two operators h_1 and x, a hamiltonian h_2 which has (almost) the same spectrum of h_1 and whose respective eigenstates are related by the intertwining operator (IO) x. More precisely, calling $\sigma(h_j)$, $j = 1, 2$, the set of eigenvalues of h_j, we find that $\sigma(h_2) \subseteq \sigma(h_1)$. These results extend what was discussed in the previous literature on this subject [5], and have the advantage of proposing a constructive procedure: while in [5] the existence of h_1, h_2 and of an operator x satisfying the intertwining condition $h_1x = xh_2$ is assumed, in [1–4] we explicitly construct h_2 from h_1 and x in such a way that h_2 satisfies a *weak form* of $h_1x = xh_2$. Moreover, as mentioned above, $\sigma(h_2) \subseteq \sigma(h_1)$ and the eigenvectors are related in a *standard* way: if $h_1\varphi_n^{(1)} = \epsilon_n\varphi_n^{(1)}$ then, if $x^\dagger\varphi_n^{(1)} \neq 0$, $h_2\left(x^\dagger\varphi_n^{(1)}\right) = \epsilon_n\left(x^\dagger\varphi_n^{(1)}\right)$, see [1–4]. It is well known that this procedure is strongly related to the supersymmetric quantum mechanics, see [6] and [7] for two rather complete overviews.

In [8–10] we have also discussed some relations between IO and the so-called *pseudo-Hermitian quantum mechanics* [11, 12] and [13] for a review, in connection with the so-called pseudo-bosons, which are excitations arising from a deformation of the canonical commutation relation. In particular, in [8] the role of Riesz bases appeared clearly, and the operators intertwined by x were not, in general, self-adjoint. This has suggested an extension of our previous results to the situation in which the hamiltonian h_1 is not self-adjoint but is rather, for instance, pseudo-Hermitian. This is discussed in section 3, which follows a section dedicated to some mathematical aspects of the self-adjoint situation. Our conclusions are contained in section 4. It should be mentioned that the analysis of IO for non-self-adjoint operators was already considered, more on a physical side, in [14], and recently in [15]. In none of these papers, however, the role of Riesz bases was considered.

2. Some mathematical aspects of the IOs

Let h_1 be a self-adjoint hamiltonian on the Hilbert space $\mathcal{H}$, $h_1 = h_1^\dagger$, whose normalized eigenvectors, $\varphi_n^{(1)}$, satisfy the following equation: $h_1\varphi_n^{(1)} = \epsilon_n\varphi_n^{(1)}$, $n \in \mathcal{I}_1 \subseteq \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let us now consider an operator x on $\mathcal{H}$ such that, calling $N_1 := xx^\dagger$ and $N_2 := x^\dagger x$, the following commutation rule (to be considered in the sense of unbounded operators, in general) is satisfied: $[N_1, h_1] = 0$. Both N_1 and N_2 are positive operators, but they could have zero in their spectra. If this is the case, then the related N_j is not invertible. Since h_1 and N_1 commute, they can be diagonalized simultaneously. Hence, it is natural to assume that the $\varphi_n^{(1)}$'s are also eigenstates of N_1. Summarizing we have

$$h_1\varphi_n^{(1)} = \epsilon_n\varphi_n^{(1)}, \qquad N_1\varphi_n^{(1)} = \nu_n\varphi_n^{(1)}, \tag{2.1}$$

for all $n \in \mathcal{I}_1$. We call $\mathcal{F}_1 = \left\{\varphi_n^{(1)}, n \in \mathcal{I}_1\right\}$ the set of these states. A second natural working assumption concerns the nature of $\mathcal{F}_1$ which is assumed here to be complete in $\mathcal{H}$ and orthonormal:

$$\left\langle\varphi_n^{(1)}, \varphi_m^{(1)}\right\rangle = \delta_{n,m}, \qquad \sum_{n\in\mathcal{I}_1} P_n^{(1)} = 1, \tag{2.2}$$

where 1 is the identity in $\mathcal{H}$ and we have introduced the following operators:

$$P_{n,m}^{(1)} f = \left\langle\varphi_n^{(1)}, f\right\rangle\varphi_m^{(1)}, \qquad \text{and} \qquad P_n^{(1)} := P_{n,n}^{(1)}, \tag{2.3}$$

for all $f \in \mathcal{H}$. It is well known that the $P_n^{(1)}$'s are orthogonal projectors: $P_n^{(1)} P_m^{(1)} = \delta_{n,m} P_n^{(1)}$ and $\left(P_n^{(1)}\right)^\dagger = P_n^{(1)}$. It is also known that the orthogonality of the vectors in $\mathcal{F}_1$ is automatic if the eigenvalues of h_1 are all different: $\epsilon_n \neq \epsilon_m$, $\forall n \neq m$. Under our assumptions we can write h_1 and N_1 as

$$h_1 = \sum_{n\in\mathcal{I}_1} \epsilon_n P_n^{(1)}, \qquad N_1 = \sum_{n\in\mathcal{I}_1} \nu_n P_n^{(1)}. \tag{2.4}$$

Moreover, since $N_1 \geqslant 0$, all its eigenvalues are non-negative: $\nu_n \geqslant 0$, $\forall n \in \mathcal{I}_1$. Of course $N_2 \geqslant 0$ as well, and since they are related by the commutator $[x, x^\dagger]$, $N_1 = N_2 + [x, x^\dagger]$, we see that, if $[x, x^\dagger] > 0$, then $N_1 > 0$. If, on the contrary, $[x, x^\dagger] < 0$, then $N_2 > 0$.

Hence, the invertibility of N_1 or N_2 can be established if $[x, x^\dagger]$ is strictly positive or negative defined. For instance, if $x = a$, where $[a, a^\dagger] = 1$, it is clear that $N_1 = a\,a^\dagger > 0$.

Remark. This is not the only case in which we can deduce the existence of N_j^{-1}. For instance, if $x = a$ with a satisfying the modified commutation relation $[a, a^\dagger]_q := a\,a^\dagger - q\,a^\dagger\,a = 1$, $q \in [0, 1]$, then, since $N_1 = [a, a^\dagger]_q + q\,N_2$, we find that $N_1 > 0$.

Let us now define the following vectors:

$$\varphi_n^{(2)} := x^\dagger\varphi_n^{(1)}, \tag{2.5}$$

for $n \in \mathcal{I}_1$. It may happen that for some n in $\mathcal{I}_1$ the action of $x^\dagger$ on $\varphi_n^{(1)}$ returns the zero vector. This means that $\ker(x^\dagger)$ is non-trivial. Of course, if $\varphi_{n_0}^{(1)} \in \ker(x^\dagger)$, then $N_1\varphi_{n_0}^{(1)} = 0$ so that $\nu_{n_0} = 0$ and the operator N_1 is not invertible. Vice versa, if N_1 is not invertible, then $\ker(N_1)$ contains some non-zero vectors. Let Ψ be such a vector. Then $N_1\Psi = 0$ and, consequently, $\|x^\dagger\Psi\|^2 = 0$. Hence, $x^\dagger\Psi = 0$, which means that $\Psi \in \ker(x^\dagger)$. In other words, $\Psi \in \ker(N_1)$ if and only if $\Psi \in \ker(x^\dagger)$.

Let now $\mathcal{I}_2 = \{n \in \mathcal{I}_1 : \varphi_n^{(2)} \neq 0\}$, and let $\mathcal{F}_2 = \{\varphi_n^{(2)}, n \in \mathcal{I}_2\}$. Of course, if $\ker(x^\dagger) = \{0\}$, then $\mathcal{I}_1 = \mathcal{I}_2$, but in general we only have the inclusion $\mathcal{I}_2 \subseteq \mathcal{I}_1$.

The set $\mathcal{F}_2$ could consist, at least in principle, of very few vectors if compared with $\mathcal{F}_1$. So the problem of completeness of $\mathcal{F}_2$ in $\mathcal{H}$ arises. Moreover, if x is not unitary (or at least isometric), we do not know if different $\varphi_n^{(2)}$'s are orthogonal to each other or if they are, by chance, eigenstates of some interesting operator. This is exactly what happens. In particular we can prove the following

Proposition 1. *Under the above hypotheses the $\varphi_n^{(2)}$'s satisfy the eigenvalue equation*

$$N_2\varphi_n^{(2)} = \nu_n\varphi_n^{(2)}, \tag{2.6}$$

for all $n \in \mathcal{I}_2$. Moreover, if for $n, m \in \mathcal{I}_2$, $n \neq m$, $\nu_n \neq \nu_m$, then $\langle\varphi_n^{(2)}, \varphi_m^{(2)}\rangle = 0$. Finally, the set $\mathcal{F}_2$ is complete in $\mathcal{H}$ if and only if N_2 is invertible.

Proof. Since $n \in \mathcal{I}_2$ the vector $\varphi_n^{(1)}$ does not belong to $\ker(x^\dagger)$, so that $\varphi_n^{(2)} \neq 0$. We have

$$N_2\varphi_n^{(2)} = (x^\dagger x)\big(x^\dagger\varphi_n^{(1)}\big) = x^\dagger N_1\varphi_n^{(1)} = \nu_n\big(x^\dagger\varphi_n^{(1)}\big) = \nu_n\varphi_n^{(2)}$$

Then our second claim is straightforward.

Let us now assume that N_2^{-1} exists. Then $\mathcal{F}_2$ is complete. Indeed, let f be an element of $\mathcal{H}$ such that $\langle f, \varphi_n^{(2)}\rangle = 0$ for all $n \in \mathcal{I}_2$. Hence, for these values of n, we also have $\langle xf, \varphi_n^{(1)}\rangle = 0$.

We consider separately two cases: $\mathcal{I}_2 = \mathcal{I}_1$ and $\mathcal{I}_2 \subset \mathcal{I}_1$. In the first case, since $\mathcal{F}_1$ is complete by assumption, we conclude that $xf = 0$, which also implies that $x^\dagger xf = N_2 f = 0$. But, since N_2 is invertible by assumption, $f = 0$. Hence, $\mathcal{F}_2$ is complete.

Suppose now that $\mathcal{I}_2 \subset \mathcal{I}_1$. Then equality $\langle xf, \varphi_n^{(1)}\rangle = 0$ for all $n \in \mathcal{I}_2$, implies that the vector xf can be written as a linear combination (which could involve infinite elements but which is clearly convergent) of the form $xf = \sum_{k\in\Gamma}\alpha_k\varphi_k^{(1)}$, where $\Gamma = \mathcal{I}_1 \setminus \mathcal{I}_2$ and the α_k's are complex constants. Taking the scalar product of both sides of this expansion for $\varphi_l^{(1)}$, $l \in \Gamma$, since for these values of l $\alpha_l = \langle\varphi_l^{(1)}, xf\rangle = \langle x^\dagger\varphi_l^{(1)}, f\rangle = 0$, we deduce that $\alpha_l = 0$ for all $l \in \Gamma$. Hence, $xf = 0$ and, as before, since N_2^{-1} does exist, $f = 0$.

To prove the inverse implication we will show that if N_2^{-1} does not exist, then $\mathcal{F}_2$ is not complete. Indeed, since N_2^{-1} does not exist there exists a vector $g \in \mathcal{H}$, $g \neq 0$, such that $N_2\, g = 0$. This implies that $0 = \langle g, N_2 g\rangle = \|xg\|^2$, so that $xg = 0$. Suppose now that $\mathcal{F}_2$ is complete. Using the equality $xg = 0$ we deduce that, for all $n \in \mathcal{I}_2$, $\langle g, \varphi_n^{(2)}\rangle = \langle xg, \varphi_n^{(1)}\rangle = 0$. Hence, due to our assumption on $\mathcal{F}_2$, we would have $g = 0$ which is against our original hypothesis. □

It is interesting to note that, while it may happen that $x^\dagger\varphi_n^{(1)} = 0$ for some $n \in \mathcal{I}_1$, it never happens that $x\,\varphi_n^{(2)} = 0$ for any $n \in \mathcal{I}_2$. This fact, which is clearly due to the procedure we are adopting, follows from the following considerations.

First we remark that, since $\forall n \in \mathcal{I}_2$, $\varphi_n^{(2)} \neq 0$, then

$$0 \neq \|\varphi_n^{(2)}\|^2 = \langle x^\dagger\varphi_n^{(1)}, x^\dagger\varphi_n^{(1)}\rangle = \langle\varphi_n^{(1)}, N_1\varphi_n^{(1)}\rangle = \nu_n.$$

Hence, $\nu_n > 0$ $\forall n \in \mathcal{I}_2$. But, for these n's, we also have $x\,\varphi_n^{(2)} = xx^\dagger\,\varphi_n^{(1)} = N_1\,\varphi_n^{(1)} = \nu_n\,\varphi_n^{(1)}$. Then, as stated, $x\varphi_n^{(2)} \neq 0$ and moreover

$$\varphi_n^{(1)} = \frac{1}{\nu_n}x\varphi_n^{(2)} = \frac{1}{\|\varphi_n^{(2)}\|^2}x\varphi_n^{(2)}. \tag{2.7}$$

The normalized vectors associated with $\mathcal{F}_2$ are therefore $\hat{\mathcal{F}}_2 = \{\hat\varphi_n^{(2)} = \frac{1}{\sqrt{\nu_n}}\,\varphi_n^{(2)}, n \in \mathcal{I}_2\}$. By means of proposition 1 above we deduce that each $\hat\varphi_n^{(2)}$ is an eigenstate of N_2, that they form an orthonormal set, at least if all the ν_n's are different, and that $\hat{\mathcal{F}}_2$ is complete if and only if N_2 is invertible. In analogy with what we have done before, let us define the operators $P_n^{(2)}$ and $\hat{P}_n^{(2)}$ as follows: $P_n^{(2)} f = \langle \varphi_n^{(2)}, f \rangle \varphi_n^{(2)}$ and $\hat{P}_n^{(2)} f = \langle \hat{\varphi}_n^{(2)}, f \rangle \hat{\varphi}_n^{(2)}$. It may be worth remarking that while the $\hat{P}_n^{(2)}$'s are orthogonal projections, the $P_n^{(2)}$'s are not, since $\left(P_n^{(2)}\right)^2 \neq P_n^{(2)}$. It is also possible to prove the following

Corollary 2. *Let us assume that for $n, m \in \mathcal{I}_2$, $n \neq m$, $\nu_n \neq \nu_m$, and that N_2^{-1} exists. Hence*

$$N_2 = \sum_{k \in \mathcal{I}_2} P_n^{(2)} = \sum_{k \in \mathcal{I}_2} \nu_n \hat{P}_n^{(2)}, \tag{2.8}$$

The proof, which makes use of the resolution of the identity for $\hat{\mathcal{F}}_2$, is trivial and will not be given here. More results on $\ker(x^\dagger)$ are discussed, under generalized assumptions, in the next section.

As in [1–4] we now define $h_2 = N_2^{-1}(x^\dagger h_1 x)$. We know that $h_2 \varphi_n^{(2)} = \epsilon_n \varphi_n^{(2)}$, for all $n \in \mathcal{I}_2$. Moreover, among other properties, we also know that $h_2 = h_2^\dagger$ if and only if $h_1 = h_1^\dagger$. Using proposition 1, which applies since we are here assuming that N_2^{-1} does exist, we deduce that $\mathcal{F}_2$ is complete so that h_2 can be written as

$$h_2 = \sum_{n=0}^{\infty} \epsilon_n \hat{P}_n^{(2)} = \sum_{n=0}^{\infty} \frac{\epsilon_n}{\nu_n} P_n^{(2)}. \tag{2.9}$$

Adopting the Dirac bra-ket notation we find that $x P_n^{(2)} = \nu_n \left|\varphi_n^{(1)}\right\rangle\left\langle\varphi_n^{(2)}\right|$ and $P_n^{(1)} x = \left|\varphi_n^{(1)}\right\rangle\left\langle\varphi_n^{(2)}\right|$ for all n. Then it follows that $x h_2 = h_1 x$, which means that x intertwines between h_2 and h_1, as in the standard papers on this subject. We will recover this intertwining relation in the following section, generalizing our previous results [1–4].

Example. *The ubiquitous harmonic oscillator*. Many aspects of our procedure can be quite well illustrated by means of the canonical commutation relation $[a, a^\dagger] = 1$ arising from the hamiltonian of a quantum harmonic oscillator, which in suitable units and putting to zero the ground state energy is $h_1 = a^\dagger a$. If $\varphi_0^{(1)}$ is such that $a\varphi_0^{(1)} = 0$, then the eigenstates of h_1 are the usual ones: $\varphi_n^{(1)} = \frac{1}{\sqrt{n!}} (a^\dagger)^n \varphi_0^{(1)}$, whose related set $\mathcal{F}_1$ is orthonormal and complete in $\mathcal{H}$. Let us now take $x = a^\dagger$. Hence, $N_1 = x x^\dagger = h_1$ and $N_2 = x^\dagger x = a a^\dagger$. Hence, N_2 is invertible and $[h_1, N_1] = 0$. Moreover, we see that $\mathcal{I}_1 = 0, 1, 2, \ldots$ while, since $x^\dagger \varphi_0^{(1)} = a\varphi_0^{(1)} = 0$, $\mathcal{I}_2 = 1, 2, \ldots$. Hence, $\mathcal{I}_2 \subset \mathcal{I}_1$. Nevertheless, the set $\hat{\mathcal{F}}_2$ coincides exactly with $\mathcal{F}_1$. Hence, it is complete in $\mathcal{H}$, as expected because of proposition 1. In this case we find easily that $h_2 = N_2^{-1}(a^\dagger h_1 a) = a a^\dagger = h_1 + 1$.

If on the other hand we take $x = a$, then $N_2 = x^\dagger x = a^\dagger a = h_1$ which is not invertible. This is in agreement with the fact that the set $\hat{\mathcal{F}}_2$ is now a proper subset of $\mathcal{F}_1$ since $\varphi_0^{(1)}$ belongs to $\mathcal{F}_1$ but not to $\hat{\mathcal{F}}_2$. Hence, $\hat{\mathcal{F}}_2$ is not complete in $\mathcal{H}$, as expected because of proposition 1. Moreover we find $h_2 = h_1 - 1$, whose eigenvalues are non-negative since its eigenvectors are $\{\varphi_1^{(1)}, \varphi_2^{(1)}, \varphi_3^{(1)}, \ldots\}$.

Example. *The deformed harmonic oscillator: quons*. Following [4] we consider two operators, B and $B^\dagger$, which satisfy the modified commutation relation $[B, B^\dagger]_q := B, B^\dagger - q B^\dagger B = 1$, $q \in [0, 1]$. Let $\varphi_0^{(1)}$ be the vacuum of B: $B\varphi_0^{(1)} = 0$. Let furthermore $h_1 = B^\dagger B$. Then, putting

$$\varphi_n^{(1)} = \frac{1}{\beta_0 \cdots \beta_{n-1}} {B^\dagger}^n \varphi_0^{(1)} = \frac{1}{\beta_{n-1}} B^\dagger \varphi_{n-1}^{(1)}, \qquad n \geqslant 1, \tag{2.10}$$

we have $h_1 \varphi_n^{(1)} = \epsilon_n \varphi_n^{(1)}$, with $\epsilon_0 = 0$, $\epsilon_1 = 1$ and $\epsilon_n = 1 + q + \cdots + q^{n-1}$ for $n \geqslant 1$. Also, the normalization is found to be $\beta_n^2 = 1 + q + \cdots + q^n$, for all $n \geqslant 0$. Hence, $\epsilon_n = \beta_{n-1}^2$ for all $n \geqslant 1$. The set of the $\varphi_n^{(1)}$'s spans the Hilbert space $\mathcal{H}$ and they are mutually orhonormal: $\langle \varphi_n^{(1)}, \varphi_k^{(1)} \rangle = \delta_{n,k}$.

We now take, as in the previous example, $x = B^\dagger$. Then $N_1 = B^\dagger B$ and $N_2 = BB^\dagger$ and, obviously, $[h_1, N_1] = 0$. Moreover, since $N_2 = BB^\dagger = 1 + q\, B^\dagger B$, and since $B^\dagger B$ is a positive operator, N_2^{-1} exists. We easily find that $h_2 := N_2^{-1}(x^\dagger h_1 x) = q\, h_1 + 1$ while

$$\varphi_n^{(2)} = B\varphi_n^{(1)} = \begin{cases} 0 & \text{if} \quad n = 0 \\ \beta_{n-1}\varphi_{n-1}^{(1)} & \text{if} \quad n \geqslant 1. \end{cases}$$

Then $\hat{\mathcal{F}}_2$ coincides again with $\mathcal{F}_1$, and so it is complete in $\mathcal{H}$, as expected because of proposition 1. If we rather take $x = B$ it is not hard to check that completeness is lost because $\varphi_0^{(1)}$ does not belong to $\hat{\mathcal{F}}_2$. This is in agreement with the fact that, since $N_2\varphi_0^{(1)} = 0$, N_2 is not invertible. Incidentally we observe that $h_2 = \frac{1}{q}(h_1 - 1)$.

3. Losing self-adjointness

In the previous section we have considered the case in which the two operators h_1 and h_2 related by the intertwining operator x are self-adjoint. Now we will remove this assumption and we will discuss some interesting consequences of this more general situation. In particular we will see that, in this new context, there are strong indications which suggest to replace o.n. bases by Riesz bases.

Let Θ_1 be a non-necessarily self-adjoint operator on $\mathcal{H}$ which admits a set $\mathcal{F}_1 = \{\varphi_n^{(1)}, n \geqslant 0\}$ of eigenstates:

$$\Theta_1\varphi_n = \epsilon_n\varphi_n, \qquad n \geqslant 0, \tag{3.1}$$

for some (in general complex) ϵ_n. In this section we will always work under the simplifying assumption that all these eigenvalues have multiplicity 1. This is useful to simplify the formulation of our results, but it could be avoided most of the times. However, examples of this situation are discussed, for instance, in [8, 9] and references therein. A class of new examples generalizing the so-called Landau levels and in which the multiplicity of each energetic level is infinity will be discussed in a paper which is now in preparation [16]. As in the previous section, we will assume now that an operator x exists, acting on $\mathcal{H}$, such that, calling $N_1 = xx^\dagger$ and $N_2 = x^\dagger x$, N_1 commutes (in the sense of unbounded operators, if needed) with Θ_1 and that N_2 is invertible. Depending on the fact that x is invertible by itself or not, we introduce two apparently different operators:

$$\Theta_2 = \begin{cases} x^{-1}\Theta_1 x, & \text{if} \quad x^{-1} \text{ exists;} \\ N_2^{-1}(x^\dagger\Theta_1 x), & \text{otherwise.} \end{cases} \tag{3.2}$$

To distinguish between these two we will sometimes call in the following $\Theta_2^{(\alpha)} = x^{-1}\Theta_1\, x$ and $\Theta_2^{(\beta)} = N_2^{-1}(x^\dagger\Theta_1 x)$. It is clear that, when x^{-1} exists, $\Theta_2^{(\beta)}$ coincides with $\Theta_2^{(\alpha)}$. However, when x^{-1} does not exist, $\Theta_2^{(\alpha)}$ makes no sense but we can still introduce $\Theta_2^{(\beta)}$. When our statements apply both for $\Theta_2^{(\alpha)}$ and for $\Theta_2^{(\beta)}$, to simplify the notation we just use Θ_2.

Remark. Analogously to what discussed in the previous section, the existence of an operator x satisfying $[xx^\dagger, \Theta_1] = 0$ has interesting consequences concerning the possibility of finding a second operator Θ_2 with (almost) the same eigenvalues (real or complex, now it doesn't matter) as Θ_1, see below.

We define, as usual,

$$\varphi_n^{(2)} = x^\dagger\varphi_n^{(1)}, \qquad n \geqslant 0, \tag{3.3}$$

which, if $\varphi_n^{(1)} \notin \ker(x^\dagger)$, is an eigenstate of Θ_2 with eigenvalue ϵ_n, independently of whether Θ_1 is self-adjoint or not. Using $[N_1, \Theta_1] = 0$ it is in fact quite easy to check that, if $\varphi_n^{(2)} \neq 0$, $\Theta_2\varphi_n^{(2)} = \epsilon_n\varphi_n^{(2)}$, for all $n \geqslant 0$.

Remark. It should be mentioned that, if x is invertible, we could also define the eigenstates of $\Theta_2^{(\alpha)}$ as $\tilde{\varphi}_n^{(2)} = x^{-1}\varphi_n^{(1)}$, which seems more appropriate since nothing should be required to the commutator between N_1 and Θ_1, at least as far as the eigenvalue equation for $\Theta_2^{(\alpha)}$ is concerned. But, in our previous papers [1–4], we have focused our attention on the situation in which x^{-1} does not necessarily exist, and this will be our main interest also here. We will return to this aspect later on.

Let us now go back for a moment to the requirement $\varphi_n^{(1)} \notin \ker(x^\dagger)$ above. It is possible to prove the following result, which extends what already stated in section 2.

Lemma 3. *With the above definitions, for a given* $n \geqslant 0$, $\varphi_n^{(1)} \in \ker(x^\dagger)$ *if and only if* $\varphi_n^{(1)} \in \ker(N_1)$ *or, equivalently, if and only if* $\varphi_n^{(2)} \in \ker(x)$.

Proof. We only prove here that if $\varphi_n^{(2)} \in \ker(x)$ then $\varphi_n^{(1)} \in \ker(x^\dagger)$. Indeed our assumption implies that $0 = x\varphi_n^{(2)} = xx^\dagger\varphi_n^{(1)}$, so that $\varphi_n^{(1)} \in \ker(N_1)$ which in turns implies, using the first statement of this lemma, that $\varphi_n^{(1)} \in \ker(x^\dagger)$. □

From the definition of $\Theta_2^{(\alpha)}$ it is clear that x is an intertwining operator, since $x\Theta_2^{(\alpha)} = \Theta_1 x$. What is not evident is whether $x\Theta_2^{(\beta)} = \Theta_1 x$ is also true. We have already considered this problem in the previous section. It is not hard to see that the answer is affirmative also in the present situation. This is a consequence of the fact that $[x^\dagger\Theta_1 x, N_2] = \left[x^\dagger\Theta_1 x, N_2^{-1}\right] = 0$, which can be proved easily. A detailed analysis produces, other than these, the following commutation rules

$$\left[\Theta_2^{(j)}, N_2\right] = \left[\Theta_2^{(j)}, N_2^{-1}\right] = \left[\left(\Theta_2^{(j)}\right)^\dagger, N_2\right] = \left[\left(\Theta_2^{(j)}\right)^\dagger, N_2^{-1}\right] = 0,$$

as well as

$$\left[\Theta_1^\dagger, N_1\right] = \left[x^\dagger\Theta_1^\dagger x, N_2\right] = \left[x^\dagger\Theta_1^\dagger x, N_2^{-1}\right] = \left[x\left(\Theta_2^{(j)}\right)^\dagger x^\dagger, N_1\right] = 0, \qquad j = \alpha, \beta,$$

and the following intertwining relations, all arising from our assumptions and from (3.1) and (3.2):

$$x\Theta_2^{(j)} = \Theta_1 x, \qquad \Theta_2^{(j)} x^\dagger = x^\dagger\Theta_1, \qquad j = \alpha, \beta. \tag{3.4}$$

Of course, the second equality in (3.4) is just the adjoint of the first one only if Θ_1 and $\Theta_2^{(j)}$ are self-adjoint, otherwise they are different. An interesting consequence is deduced if $\Theta_2 = \Theta_1^\dagger$, which is important, as we will discuss in the following, in the context of pseudo-Hermitian quantum mechanics (PHQM) [11–13]. In this case x is not only an IO but it also commutes with $\Theta_1 + \Theta_1^\dagger$, as well as $x^\dagger$ does. Hence, N_1, N_2 and $\Theta_1 + \Theta_2$ are three self-adjoint operators such that $[N_1, \Theta_1 + \Theta_2] = [N_2, \Theta_1 + \Theta_2] = 0$, but, in general $[N_1, N_2] \neq 0$. So they are not expected to admit a set of common eigenvectors.

An evident difference between N_1 and N_2 is that, while N_2 is strictly positive by assumption, N_1 needs not to be invertible. On the other hand our original assumption $[N_1, \Theta_1] = 0$ is reflected by $[N_2, \Theta_2]$, which is also zero. However, analogously to what we observed in the previous section, if x is such that $[x, x^\dagger] \geqslant 0$ (in the sense of the operators), then $N_1 = [x, x^\dagger] + N_2$ is also strictly positive so that it is invertible. If this is the case, the commutation rules listed before can be enriched by other rules involving N_1^{-1}, which will play no role here, and therefore will not be considered. More interesting is the following.

Lemma 4. *If N_2^{-1} exists, $[\Theta_1, N_1] = 0$ and if (3.1) holds, then $\Theta_2 = \Theta_2^\dagger$ if and only if $\Theta_1 = \Theta_1^\dagger$.*

Proof. We give two different proofs for $\Theta_2^{(\alpha)}$ and $\Theta_2^{(\beta)}$.

Let us first suppose that $\Theta_2^{(\alpha)} = \left(\Theta_2^{(\alpha)}\right)^\dagger$. This means that we are working under the assumption that x^{-1} does exist. Hence, $x^{-1}\Theta_1 x = x^\dagger\Theta_1^\dagger(x^{-1})^\dagger$, which is equivalent to $\Theta_1 = N_1\Theta_1^\dagger N_1^{-1}$. This equality, since $\left[N_1, \Theta_1^\dagger\right] = 0$, implies that $\Theta_2^{(\alpha)} = \left(\Theta_2^{(\alpha)}\right)^\dagger$ if and only if $\Theta_1 = \Theta_1^\dagger$.

Let us now suppose that $\Theta_2^{(\beta)} = \left(\Theta_2^{(\beta)}\right)^\dagger$ (we are no longer requiring x to be invertible). This is equivalent to $N_2^{-1}(x^\dagger\Theta_1 x) = \left(x^\dagger\Theta_1^\dagger x\right)N_2^{-1}$, which, since $\left[x^\dagger\Theta_1 x, N_2^{-1}\right] = 0$, is equivalent to $x^\dagger\Theta_1 x = x^\dagger\Theta_1^\dagger x$. Now, using (3.4), this can be rewritten as $\Theta_2^{(\alpha)}x^\dagger x = \left(\Theta_2^{(\alpha)}\right)^\dagger x^\dagger x$ which can be multiplied from the right by N_2^{-1}, giving back $\Theta_2^{(\alpha)} = \left(\Theta_2^{(\alpha)}\right)^\dagger$. Hence, $\Theta_2^{(\beta)} = \left(\Theta_2^{(\beta)}\right)^\dagger$ if and only if $\Theta_2^{(\alpha)} = \left(\Theta_2^{(\alpha)}\right)^\dagger$ which, in turns, is equivalent to $\Theta_1 = \Theta_1^\dagger$. □

Proposition 5. *Under the assumptions of lemma 4, let us suppose that, for a fixed $k \geqslant 0$, we have $\varphi_k^{(1)} \notin \ker(N_1)$. Then $\varphi_k^{(1)}$ is also eigenstate of N_1 with a strictly positive eigenvalue ν_k. Moreover $\varphi_k^{(2)}$ is a non-zero eigenstate of N_2 with eigenvalue ν_k.*

Furthermore, if ν_k has multiplicity one, $m(\nu_k) = 1$, then $\varphi_k^{(1)}$ is also eigenstate of $\Theta_1^\dagger$ with eigenvalue $\overline{\epsilon_k}$, and $\varphi_k^{(2)}$ is an eigenstate of $\Theta_2^\dagger$ with eigenvalue $\overline{\epsilon_k}$.

Finally, if $m(\nu_k) = 1$ for all $k \geqslant 0$, then $\mathcal{F}_1$ and $\mathcal{F}_2$ are orthogonal systems in $\mathcal{H}$. Moreover:

- *if $\mathcal{F}_1$ is complete in $\mathcal{H}$, then $\left[\Theta_1, \Theta_1^\dagger\right] = 0$;*
- *if $\mathcal{F}_2$ is complete in $\mathcal{H}$, then $\left[\Theta_2, \Theta_2^\dagger\right] = 0$.*

Proof. Since $[\Theta_1, N_1] = 0$, and since $m(\epsilon_k) = 1$, it follows that $N_1\varphi_k^{(1)}$ must be proportional to $\varphi_k^{(1)}$ itself. Let ν_k be this proportionality constant. Hence, $N_1\varphi_k^{(1)} = \nu_k\varphi_k^{(1)}$ and $\nu_k = \frac{\|x^\dagger\varphi_k^{(1)}\|^2}{\|\varphi_k^{(1)}\|^2}$, which is strictly positive since $\varphi_k^{(1)} \notin \ker(x^\dagger)$, see lemma 3. This lemma also implies that $\varphi_k^{(2)} \neq 0$, and we have $N_2\varphi_k^{(2)} = x^\dagger x x^\dagger\varphi_k^{(1)} = x^\dagger N_1\varphi_k^{(1)} = \nu_k\varphi_k^{(2)}$.

Now, we note that $\varphi_k^{(1)} \notin \ker\left(\Theta_1^\dagger\right)$ and that $\varphi_k^{(2)} \notin \ker\left(\Theta_2^\dagger\right)$. Indeed we have

$$\left\langle\varphi_k^{(1)}, \Theta_1^\dagger\varphi_k^{(1)}\right\rangle = \left\langle\Theta_1\varphi_k^{(1)}, \varphi_k^{(1)}\right\rangle = \overline{\epsilon_k}\|\varphi_k^{(1)}\|^2,$$

and analogously $\left\langle\varphi_k^{(2)}, \Theta_2^\dagger\varphi_k^{(2)}\right\rangle = \overline{\epsilon_k}\left\|\varphi_k^{(2)}\right\|^2$, which are both different from zero.

If we now assume that $m(\nu_k) = 1$, since $\left[N_1, \Theta_1^\dagger\right] = 0$, we conclude that $\Theta_1^\dagger\varphi_k^{(1)}$ is proportional to $\varphi_k^{(1)}$ itself, and the proportionality constant is easily found to be $\overline{\epsilon_k}$: $\Theta_1^\dagger\varphi_k^{(1)} = \overline{\epsilon_k}\varphi_k^{(1)}$. In a similar way we also deduce that $\Theta_2^\dagger\varphi_k^{(2)} = \overline{\epsilon_k}\varphi_k^{(2)}$.

Finally, let us assume that $m(\nu_k) = 1$ for all $k \geqslant 0$. This implies that, taking $j = 1, 2$, since different $\varphi_k^{(j)}$'s are eigenvectors of self-adjoint operators N_j corresponding to different eigenvalues, they must be orthogonal:

$$\left\langle\varphi_k^{(j)}, \varphi_n^{(j)}\right\rangle = 0$$

if $k \neq n$, for $j = 1, 2$. Moreover, if for instance $\mathcal{F}_1$ is complete in $\mathcal{H}$, hence it is an o.n. basis. Therefore, our last claim easily follows from the fact that $\Theta_1\Theta_1^\dagger\varphi_k^{(1)} = \Theta_1^\dagger\Theta_1\varphi_k^{(1)} = |\epsilon_k|^2\varphi_k^{(1)}$, for all $k \geqslant 0$. □

3.1. The role of Riesz bases

Since Θ_k, $k = 1, 2$, are not, in general, self-adjoint operators, the sets of their eigenstates $\mathcal{F}_1$ and $\mathcal{F}_2$ are not orthonormal, in general. This is one of the reasons why we are now considering the role of Riesz bases in the present context. The second reason, as already stated in the introduction, is that in a series of recent papers [8–10], Riesz bases have already appeared in analogous problems, and they have shown to be quite relevant.

Let us now assume that the set $\mathcal{F}_1$ of eigenstates of Θ_1 is a Riesz basis for $\mathcal{H}$. This means that a bounded operator T exists, with bounded inverse T^{-1}, and an o.n. basis $\mathcal{E} = \{e_n \in \mathcal{H},\ n \geqslant 0\}$, such that $\varphi_n^{(1)} = Te_n$ for all n. Equivalently, from [17, 18], we can say that the vectors of $\mathcal{F}_1$ are linearly independent and two constants exist, $0 < A \leqslant B < \infty$, such that, for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leqslant \sum_{n\geqslant 0} \left|\left\langle\varphi_n^{(1)}, f\right\rangle\right|^2 \leqslant B\|f\|^2.$$

Then a bounded operator (the *frame operator*) $S_1 := \sum_{n\geqslant 0} \left|\varphi_n^{(1)}\right\rangle\left\langle\varphi_n^{(1)}\right| = TT^\dagger$ exists, with bounded inverse, and the set $\tilde{\mathcal{F}}_1 = \left\{\tilde\varphi_n^{(1)} = S_1^{-1}\varphi_n^{(1)}\right\}$ is biorthogonal to $\mathcal{F}_1$: $\left\langle\varphi_n^{(1)}, \tilde\varphi_k^{(1)}\right\rangle = \delta_{n,k}$, for all $n, k \geqslant 0$. Moreover, $\tilde{\mathcal{F}}_1$ is a Riesz basis by itself, since $\tilde\varphi_n^{(1)} = \tilde{T}e_n$ for all n, with $\tilde{T} = S_1^{-1}T$. Indeed, $\tilde{T}$ is bounded with bounded inverse. Also, the following resolutions of the identity can be deduced:

$$\sum_{n\geqslant 0} \left|\tilde\varphi_n^{(1)}\right\rangle\left\langle\varphi_n^{(1)}\right| = \sum_{n\geqslant 0} \left|\varphi_n^{(1)}\right\rangle\left\langle\tilde\varphi_n^{(1)}\right| = 1. \tag{3.5}$$

Let now Θ_1, $\mathcal{F}_1$ and x be as in the first part of section 3. It is interesting to analyze the nature of $\mathcal{F}_2$. The first easy result is the following.

Lemma 6. *Let $\mathcal{F}_1$ be a Riesz basis. Then the following are equivalent: (a) $x = S_1^{-1}$; (b) the set $\mathcal{F}_2 = \left\{\varphi_n^{(2)} = x^\dagger\varphi_n^{(1)}, n \geqslant 0\right\}$ is a Riesz basis biorthogonal to $\mathcal{F}_1$.*

Proof. The proof that (a) implies (b) follows from our previous discussion, noting that in this case $\mathcal{F}_2 = \tilde{\mathcal{F}}_1$.

The converse implication can be proved as follows: first we note that

$$\delta_{n,k} = \left\langle\varphi_n^{(1)}, \varphi_k^{(2)}\right\rangle = \left\langle Te_n, x^\dagger Te_k\right\rangle = \left\langle T^\dagger xTe_n, e_k\right\rangle,$$

for all n, k, which implies that $T^\dagger xT = 1$. Then, recalling that T is invertible, we get $x = (TT^\dagger)^{-1} = S_1^{-1}$. □

By means of the resolutions (3.5) we can easily deduce that, if $\mathcal{F}_2 = \tilde{\mathcal{F}}_1$,

$$\Theta_1 = \sum_{n\geqslant 0} \epsilon_n \left|\varphi_n^{(1)}\right\rangle\left\langle\varphi_n^{(2)}\right|, \qquad \text{and} \qquad \Theta_2 = \sum_{n\geqslant 0} \epsilon_n \left|\varphi_n^{(2)}\right\rangle\left\langle\varphi_n^{(1)}\right|, \tag{3.6}$$

from which it is possible to deduce that $\Theta_1 = \Theta_2^\dagger$ if and only if ϵ_n is real for all n.

Of course, having two biorthogonal Riesz bases $(\mathcal{F}_1, \mathcal{F}_2)$ of eigenstates of (Θ_1, Θ_2) looks quite interesting and is a natural extension of what happens for self-adjoint operators. However, the following proposition can be seen as a sort of no-go result. Indeed it states that, under the hypotheses we are considering here, Θ_2 coincides with Θ_1 and $\mathcal{F}_2$ coincides with $\mathcal{F}_1$. In other words, we are just only apparently introducing new vectors and a new operator.

Proposition 7. *Let $\mathcal{F}_1$ be a set of eigenstates of Θ_1 which is also a Riesz basis, and let us suppose that $x = S_1^{-1}$ and that $[\Theta_1, N_1] = 0$. Hence, $[\Theta_1, x] = 0$ and $\Theta_2 = \Theta_1$. Moreover, for all $n \geqslant 0$, $\varphi_n^{(2)}$ is proportional to $\varphi_n^{(1)}$.*

Proof. First we remark that, since $x = S_1^{-1}$, and since S_1 is self-adjoint, then $N_1 = N_2 = S_1^{-2}$, which is clearly invertible. Moreover $\Theta_2^{(\alpha)}$ and $\Theta_2^{(\beta)}$ coincide, since x^{-1} exists and is equal to S_1. Now, since N_1 is a positive and bounded operator commuting with Θ_1, it is known [19] that there exists a unique positive operator, the square root of N_1, which commutes with all the operators which commute with N_1. Of course this positive square root is S_1^{-1} itself, and then it follows that $\left[\Theta_1, S_1^{-1}\right] = 0$. Our claims now easily follow. □

This result suggests that the assumptions contained in lemma 6 are too restrictive and should be weakened. This is exactly what we will do in the rest of this section. We begin with the following proposition, related to the structure of Θ_1 in connection with its pseudo-Hermiticity. This result generalizes those contained in [12].

Proposition 8. *If Θ_1 admits a basis of eigenvectors which is a Riesz basis, with real eigenvalues, then there exists an operator T, bounded with bounded inverse, such that Θ_1 is $(T\,T^\dagger)^{-1}$-pseudo-Hermitian. Vice versa, if Θ_1 is $(T\,T^\dagger)^{-1}$-pseudo-Hermitian for some operator T, bounded with bounded inverse, and if the operator $T^{-1}\Theta_1 T$ admits an o.n. basis of $\mathcal{H}$ as eigenstates, then Θ_1 admits a basis of eigenvectors which is a Riesz basis, with real eigenvalues.*

Proof. Let us first assume that Θ_1 admits a basis of eigenvectors $\mathcal{F}_1$ which is a Riesz basis, and that its eigenvalues are real: $\Theta_1\varphi_n^{(1)} = \epsilon_n\varphi_n^{(1)}$, for all n. Hence, as already stated, $\varphi_n^{(1)} = Te_n$ for a certain $T \in B(\mathcal{H})$, invertible with $T^{-1} \in B(\mathcal{H})$, and an o.n. basis $\mathcal{E} = \{e_n\}$. Hence, the eigenvalue equation for Θ_1 can be rewritten as

$$\Theta_{1,T}e_n = \epsilon_n e_n, \qquad n \geqslant 0, \tag{3.7}$$

where $\Theta_{1,T} := T^{-1}\Theta_1 T$. Of course this means that $\Theta_{1,T} = \sum_{n\geqslant 0}\epsilon_n|e_n\rangle\langle e_n|$ which, since ϵ_n is real for all n, implies that $\Theta_{1,T}$ is self-adjoint. Now, simple algebraic manipulations show that $\Theta_{1,T} = \Theta_{1,T}^\dagger$ if and only if $\Theta_1^\dagger = (T\,T^\dagger)^{-1}\Theta_1(T\,T^\dagger)$, so that Θ_1 is $(T\,T^\dagger)^{-1}$-pseudo-Hermitian [11–13].

Vice versa, let us assume that an operator T exists, bounded with inverse bounded, such that Θ_1 is $(T\,T^\dagger)^{-1}$-pseudo-Hermitian. Then $\Theta_1^\dagger = (T\,T^\dagger)^{-1}\Theta_1(T\,T^\dagger)$ which implies that, defining as before $\Theta_{1,T} := T^{-1}\Theta_1 T$, $\Theta_{1,T} = \Theta_{1,T}^\dagger$. Hence, since an o.n. basis $\mathcal{E} = \{e_n\}$ of eigenvectors of $\Theta_{1,T}$ exists by assumption, $\Theta_{1,T}\,e_n = \epsilon_n e_n$, for all n, it follows that $\epsilon_n \in \mathbb{R}$. It is further clear that, defining $\varphi_n^{(1)} = Te_n$ and $\mathcal{F}_1 = \{\varphi_n^{(1)},\ n \geqslant 0\}$, this set is a Riesz basis of eigenvectors of Θ_1, with real eigenvalues. □

A simple consequence of this proposition is the following

Corollary 9. *Let us assume that, for a certain $T \in B(\mathcal{H})$ with bounded inverse, Θ_1 is $(T\,T^\dagger)^{-1}$-pseudo-Hermitian. Let us further assume that the IO x is bounded and invertible and that $[\Theta_1, N_1] = 0$. Then $\Theta_2 = x^{-1}\Theta_1 x$ admits a Riesz basis of eigenvectors $\mathcal{F}_2 = \left\{\varphi_n^{(2)} = x^\dagger\varphi_n^{(1)},\ n \geqslant 0\right\}$ with real eigenvalues.*

The proof is straightforward and will not be given here. However, it should be mentioned that $(\mathcal{F}_1, \mathcal{F}_2)$ are not biorthogonal in general, and this makes the procedure non-trivial. Indeed we have already seen that, if $(\mathcal{F}_1, \mathcal{F}_2)$ are biorthogonal, then $x = S_1^{-1}$ necessarily and, as a consequence, $\Theta_2 = \Theta_1$.

Remark. It may be worth noting that, as widely discussed in [8], different families of coherent states, or of generalized coherent states, can be associated with any Riesz basis. In particular, we can construct two *dual* families of coherent states which, together, produce a decomposition of the identity. This aspect of the theory will not be considered here.

We conclude this section reconsidering what we have done in [8] in the following.

Our starting point is a Riesz basis of $\mathcal{H}$, $\mathcal{F} := \{\varphi_n,\ n \geqslant 0\}$. Then, if $S = \sum_n |\varphi_n\rangle\langle\varphi_n|$ is its frame operator, we define $\hat{\mathcal{F}} = \{\hat\varphi_n := S^{-1/2}\varphi_n,\ n \geqslant 0\}$. This is an o.n. basis: $\sum_n |\hat\varphi_n\rangle\langle\hat\varphi_n| = 1$ and $\langle\hat\varphi_n, \hat\varphi_k\rangle = \delta_{n,k}$. Let now A be a lowering operator defined by $A\hat\varphi_n = \sqrt{n}\hat\varphi_{n-1}$, for all $n \geqslant 0$. In particular this means that $A\varphi_0 = 0$. Its adjoint is a raising operator: $A^\dagger\hat\varphi_n = \sqrt{n+1}\hat\varphi_{n+1}$, $n \geqslant 0$, and they satisfy the canonical commutation relation $[A, A^\dagger] = 1$. Let us now define

$$a = S^{1/2}AS^{-1/2} \qquad \text{and} \qquad b = S^{1/2}A^\dagger S^{-1/2}.$$

Then $[a, b] = 1$ and, in general, $a \neq b^\dagger$. This is the commutation rule which defines the so-called pseudo-bosons [20]. Moreover $a\varphi_n = \sqrt{n}\varphi_{n-1}$ is a lowering operator for $\mathcal{F}$ while the related raising operator is b: $b\varphi_n = \sqrt{n+1}\varphi_{n+1}$ [8]. Now we put $\mathcal{F}_1 \equiv \mathcal{F}$, i.e. $\varphi_n^{(1)} = \varphi_n$ for all n, and $\mathcal{F}_2 = \{\varphi_n^{(2)} = S^{-1}\varphi_n^{(1)}\}$, and we see that $\varphi_n^{(1)} = \frac{1}{\sqrt{n!}}\, b^n\varphi_0$ and $\varphi_n^{(2)} = \frac{1}{\sqrt{n!}}\,(a^\dagger)^n\varphi_0^{(2)}$, where $\varphi_0^{(2)} = S^{-1}\varphi_0^{(1)} = S^{-1}\varphi_0$. $(\mathcal{F}_1, \mathcal{F}_2)$ are biorthogonal and, defining $\Theta_1 = ba$ and $\Theta_2 = \Theta_1^\dagger = a^\dagger b^\dagger$, we find that $\Theta_1\varphi_n^{(1)} = n\varphi_n^{(1)}$ and $\Theta_2\varphi_n^{(2)} = n\varphi_n^{(2)}$, for all $n \geqslant 0$.

Moreover, S acts as an IO since $\Theta_1 S = S\Theta_2$. But, since S is invertible, this also implies that $\Theta_2 = S^{-1}\Theta_1 S$ so that, recalling that $\Theta_2 = \Theta_1^\dagger$, Θ_1 is S^{-1}-pseudo-Hermitian.

The non-triviality of this example, i.e. the fact that $\Theta_2 \neq \Theta_1$, is based on the fact that the main assumption of proposition 7 is violated: $[\Theta_1, N_1] = [\Theta_1, S^{-2}] \neq 0$, in general. This can be understood since we can show, first of all, that $[\Theta_1, S^{-2}] = 0$ if and only if $[A^\dagger A, S^2] = 0$. But $\langle\hat\varphi_l, [A^\dagger A, S^2]\hat\varphi_n\rangle = (l-n)\langle\hat\varphi_l, S^2\hat\varphi_n\rangle$ which is zero, for all l and n, if S^2 is diagonal in $\hat{\mathcal{F}}$ but not in general. Therefore, without further assumptions, $[\Theta_1, N_1] \neq 0$.

Analogously, it is also possible to check directly that, but if S is diagonal in $\hat{\mathcal{F}}$, $\langle\hat\varphi_l, [A^\dagger A, S]\hat\varphi_n\rangle \neq 0$ and, as a consequence, $[\Theta_1, S^{-1}] \neq 0$ and, yet, $\Theta_2 \neq \Theta_1$.

4. Conclusions

We have considered some mathematical aspects of intertwining operators extending our previous results also to the situation of non-self-adjoint operators. This has produced interesting results in connection with pseudo-Hermitian quantum mechanics (PHQM). The role of Riesz bases, in the present context, has been analyzed in some detail and they appear to be relevant alternatives to o.n. bases whenever we look for eigenstates of a non-self-adjoint operator.

We end the paper with a short summary of our point of view:

In order to relate the eigensystems of Θ_1 and Θ_2 it is sufficient to have some intertwining relation $\Theta_2 x = x\Theta_1$ and it is not necessary that x^{-1} exists. Indeed, if Φ is an eigenstate of Θ_1 with eigenvalue ϵ, then $x\Phi$ is either zero or is an eigenstate of Θ_2 with the same eigenvalue.

However, if we want to talk of *standard* pseudo-Hermiticity, Θ_2 must coincide with $\Theta_1^\dagger$ and x must be invertible. But, since if x is not invertible our approach still works and produces (quasi)-isospectral operators, we believe it may be worth investigating whether some other aspects of PHQM, other than the coincidence of the eigenvalues, can be extended in our more general settings. This work, which we have just began here, is now in progress.

Acknowledgments

The author acknowledges financial support by the Murst, within the project *Problemi Matematici Non Lineari di Propagazione e Stabilità nei Modelli del Continuo*, coordinated by Prof. T Ruggeri.

References

[1] Bagarello F 2008 Extended SUSY quantum mechanics, intertwining operators and coherent states *Phys. Lett.* A **372** 6226–31

[2] Bagarello F 2009 Vector coherent states and intertwining operators *J. Phys. A: Math. Theor.* **42** 075302

[3] Bagarello F 2009 Intertwining operators between different Hilbert spaces: connection with frames *J. Math. Phys.* **50** 043509

[4] Bagarello F 2009 Quons, coherent states and intertwining operators *Phys. Lett.* A **373** 2637–42

[5] Kuru S, Tegmen A and Vercin A 2001 Intertwined isospectral potentials in an arbitrary dimension *J. Math. Phys.* **42** 3344–60

Kuru S, Demircioglu B, Onder M and Vercin A 2002 Two families of superintegrable and isospectral potentials in two dimensions *J. Math. Phys.* **43** 2133–50

Samani K A and Zarei M 2005 Intertwined hamiltonians in two-dimensional curved spaces *Ann. Phys.* **316** 466–82

[6] Cooper F, Khare A and Sukhatme U 2001 *Supersymmetry and Quantum Mechanics* (Singapore: World Scientific)

[7] Junker G 1996 *Supersymmetric Methods in Quantum and Statistical Physics* (Berlin: Springer)

[8] Bagarello F 2010 Pseudo bosons, Riesz bases and coherent states *J. Math. Phys.* **50** 023531

[9] Bagarello F 2010 Construction of pseudo bosons systems *J. Math. Phys.* at press

[10] Bagarello F 2010 Pseudo bosons arising from Riesz bases *Bollettino del Dipartimento di Metodi e Modelli Matematici* at press

[11] Mostafazadeh A 2002 Pseudo supersymmetric quantum mechanics and isospectral pseudo Hermitian Hamiltonians *Nucl. Phys.* B **640** 419–34

Mostafazadeh A and Batal A 2004 Physical aspects of pseudo Hermitian and PT-symmetric quantum mechanics *J. Phys. A: Math. Gen.* **37** 11645–79

[12] Mostafazadeh A 2002 Pseudo Hermiticity versus PT-Symmetry II: a complete characterization of non-Hermitian Hamiltonians with a real spectrum *J. Math. Phys.* **43** 2814–6

[13] Mostafazadeh A 2008 Pseudo Hermitian quantum mechanics arXiv:quant-ph/0810.5643

[14] Andrianov A A, Cannata F, Dedonder J-P and Ioffe M V 1999 SUSY quantum mechanics with complex superpotentials and real energy spectra *Int. J. Mod. Phys.* A **14** 2675–88

[15] Bazeiaa D, Das A, Greenwood L and Losano L 2009 The structure of supersymmetry in PT symmetric quantum mechanics *Phys. Lett.* B **673** 283–7

[16] Ali S T, Bagarello F and Gazeau J-P Generalized Landau levels and two-dimensional pseudo bosons (in preparation)

[17] Young R 1980 *An Introduction to Nonharmonic Fourier Series* (New York: Academic)

[18] Christensen O 2003 *An Introduction to Frames and Riesz Bases* (Boston, MA: Birkhäuser)

[19] Reed M and Simon B 1980 *Methods of Modern Mathematical Physics* vol I (New York: Academic)

[20] Trifonov D A 2009 Pseudo boson coherent and Fock states arXiv:quant-ph/0902.3744